出典
　2011 年　三重大学
ポイント
(1)　始点変換、絶対値の処理の仕方は基礎中の基礎なので解けなければならない

(2)　垂直ならば$\left(\text{内積}\right)=0$と、平面上に点が存在する条件を知っておけば解ける問題である。

(1) $$\left|\vec{a}\right|=\left|\overrightarrow{\mathrm{OA}}\right|=\sqrt{2},\ \left|\vec{b}\right|=\left|\overrightarrow{\mathrm{OB}}\right|=\sqrt{5},\ \left|\vec{c}\right|=\left|\overrightarrow{\mathrm{OC}}\right|=\sqrt{2}$$

$\left|\overrightarrow{\mathrm{AB}}\right|=\left|\overrightarrow{\mathrm{OB}}-\overrightarrow{\mathrm{OA}}\right|=\left|\vec{b}-\vec{a}\right|$より,

$$\begin{aligned}&\left|\vec{b}-\vec{a}\right|^2=3^2\\ \Leftrightarrow\ &\left|\vec{b}\right|^2-2\vec{a}\cdot\vec{b}+\left|\vec{a}\right|^2=9\\ \Leftrightarrow\ &\left(\sqrt{5}\right)^2-2\vec{a}\cdot\vec{b}+\left(\sqrt{2}\right)^2=9\\ &\therefore\ \boldsymbol{\vec{a}\cdot\vec{b}=-1}\end{aligned}$$

$\angle\mathrm{AOC}=\angle\mathrm{BOC}=\dfrac{\pi}{2}$より,

$$\therefore\ \boldsymbol{\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=0}$$

(2)　$\overrightarrow{\mathrm{OD}}=\dfrac{2}{3}\vec{a}+\dfrac{1}{3}\vec{b}$より,

$$\begin{aligned}\left|\overrightarrow{\mathrm{OD}}\right|^2&=\left|\frac{2}{3}\vec{a}+\frac{1}{3}\vec{b}\right|^2\\&=\frac{4}{9}\left|\vec{a}\right|^2+\frac{4}{9}\vec{a}\cdot\vec{b}+\frac{1}{9}\left|\vec{b}\right|^2\\&=1\\ \therefore\ \left|\overrightarrow{\mathrm{OD}}\right|&=1\end{aligned}$$

$$\overrightarrow{\mathrm{OC}}\cdot\overrightarrow{\mathrm{OD}}=\vec{c}\left(\frac{2}{3}\vec{a}+\frac{1}{3}\vec{b}\right)=0$$

$\overrightarrow{\mathrm{OH}}=\left(1-t\right)\overrightarrow{\mathrm{OD}}+t\overrightarrow{\mathrm{OC}}$とおくと，$\mathrm{OH}\perp\mathrm{CD}$より,

$$\overrightarrow{\mathrm{OH}}\cdot\overrightarrow{\mathrm{CD}}=0$$
$$\Leftrightarrow \left\{(1-t)\overrightarrow{\mathrm{OD}}+t\overrightarrow{\mathrm{OC}}\right\}\cdot\left(\overrightarrow{\mathrm{OD}}-\overrightarrow{\mathrm{OC}}\right)=0$$
$$\Leftrightarrow (1-t)\left|\overrightarrow{\mathrm{OD}}\right|^2-t\left|\overrightarrow{\mathrm{OC}}\right|^2+(2t-1)\overrightarrow{\mathrm{OC}}\cdot\overrightarrow{\mathrm{OD}}=0$$
$$\Leftrightarrow (1-t)\cdot 1^2-t\cdot\left(\sqrt{2}\right)^2+(2t-1)\cdot 0=0$$
$$\therefore\ t=\frac{1}{3}$$

よって，

$$\overrightarrow{\mathrm{OH}}=\frac{2}{3}\overrightarrow{\mathrm{OD}}+\frac{1}{3}\overrightarrow{\mathrm{OC}}$$
$$=\frac{2}{3}\left(\frac{2}{3}\vec{a}+\frac{1}{3}\vec{b}\right)+\frac{1}{3}\vec{c}$$
$$=\underline{\boldsymbol{\frac{4}{9}\vec{a}+\frac{2}{9}\vec{b}+\frac{1}{3}\vec{c}}}$$

また，

$$\left|\overrightarrow{\mathrm{OH}}\right|^2=\left|\frac{2}{3}\overrightarrow{\mathrm{OD}}+\frac{1}{3}\overrightarrow{\mathrm{OC}}\right|^2$$
$$=\frac{4}{9}\left|\overrightarrow{\mathrm{OD}}\right|^2+\frac{4}{9}\overrightarrow{\mathrm{OC}}\cdot\overrightarrow{\mathrm{OD}}+\frac{1}{9}\left|\overrightarrow{\mathrm{OC}}\right|^2$$
$$=\frac{4}{9}\cdot 1^2+\frac{4}{9}\cdot 0+\frac{1}{9}\cdot\left(\sqrt{2}\right)^2$$
$$=\frac{6}{9}$$

よって，

$$\left|\overrightarrow{\mathrm{OH}}\right|=\sqrt{\frac{6}{9}}=\underline{\boldsymbol{\frac{\sqrt{6}}{3}}}$$